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Characterizations of Weighted Besov and Triebel–Lizorkin Spaces via Temperatures

HUY QUI BUI

Department of Mathematics, Faculty of Science, Hiroshima University, Hiroshima 730 Japan

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Hardy–Littlewood type characterizations (via temperatures on a half-space) of weighted Besov and Triebel–Lizorkin spaces studied recently by the author are given. In the proof of the sufficient parts of the results, a "sub-mean-value property" of temperatures is used in an elegant way to get control of certain terms by the Hardy maximal function or its vector-valued version. As a byproduct of the results, another characterization of weighted Hardy spaces as well as characterizations of the Gauss–Weierstrass integrals on a strip domain of distributions in the above weighted spaces is obtained also.

INTRODUCTION

In a recent paper [5], we have extended a part of the theory of Besov and Triebel–Lizorkin spaces initiated by Peetre [13, 14] to the context of weighted spaces in which the weight function w is in the class A_∞. The method of Peetre, which we have adopted, is to decompose a distribution on the Fourier transform side, and a maximal function technique is essential in this approach.

On the other hand, in the classical case, i.e., when $w = 1$ and $1 \leqslant p$, $q \leqslant \infty$, the study of the Besov (–Lipschitz) spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ are usually done by considering the harmonic or thermic extensions (to R^{n+1}_+) of distributions in these spaces, and one often derives properties of $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ by studying these extensions to R^{n+1}_+ (see, e.g., [3, 8, 10, 16, 18]). We notice that the role played by the Poisson kernel or the Gauss–Weierstrass kernel is highlighted in this approach; a unified treatment of these two kernels in the $B^s_{p,q}$-case can be done with the aid of the theory of semi-groups of operators for which we refer to [1, 2].

In the more recent case $0 < p < 1$, although there seems to be no semi-group theory counterpart, the characterizations of $\dot{B}^s_{p,q}$ ($s < 1$, $q = \infty$) via harmonic functions, and those of $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ via temperatures have been

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given in [14] and [4], respectively. Our aim in this paper is to extend the characterizations of Besov spaces in [4] to the present weighted case and to derive similar results for weighted Triebel–Lizorkin spaces. We should point out here that the more interesting part of our results concerns the sufficient conditions that give criteria for a distribution to belong to these weighted spaces. The main step in the proof of this part is the use of a "sub-mean-value property" of temperatures, together with the Hardy maximal function or its vector-valued version, to get control of certain terms we wish to estimate, the consequence of which is that we can then make use of results on weighted estimates existing in the literatures (see the proofs of Theorem 1(ii) and Theorem 4(ii)).

The plan of the paper is as follows. In Section 1, we recall definitions of the spaces to be studied, and then state all of our results. In Section 2, we make some comments and remarks on known results as well as further results. Finally, the results stated in Section 1 are proved in Section 3. While the proofs of the results for nonhomogeneous spaces are given in fairly complete details, those for homogeneous spaces are only outlined since we feel that the interested reader can work out details by himself without much difficulty by modifying the arguments given in the former case.

1. Definitions and Results

1.1. *Definitions*

We follow the notation and terminologies in [5]. Some of them are recalled in this subsection, and we refer to [5] for unexplained terminologies as well as related literatures. We shall deal with (measurable) functions or (tempered) distributions defined on the n-dimensional Euclidean space R^n. *Throughout this note, we let w denote a weight function in the class A_∞ and* $r_0 = \inf\{r;\, w \in A_r\}$ (cf. [5, Sect. 1; 6]); a well known and useful subclass of functions in A_∞ is $\{w_\rho;\, w_\rho(x) = |x|^\rho,\ -n < \rho < \infty\}$. We let

$$L^p_w = \left\{ f;\, \|f\|_{p,w} = \left(\int |f(x)|^p\, w(x)\, dx \right)^{1/p} < \infty \right\}, \qquad 0 < p < \infty.$$

Here and hereafter, all integrals are assumed to be extended over all of R^n unless otherwise indicated. *We assume that the ranges of the parameters will be always as follows*: $0 < p < \infty$, $0 < q \leqslant \infty$, *and* $-\infty < s < \infty$; we do not consider the case $p = \infty$ since by our definitions in [5], weighted Besov spaces coincide with nonweighted ones when $p = \infty$, while for Triebel–Lizorkin spaces, even for $w = 1$, one can develop a good theory only in the case $p < \infty$ (see [19, 2.1.4]). Let ψ be a function in $\mathscr{S}$ such that $\operatorname{supp} \psi = \{\frac{1}{2} \leqslant |\xi| \leqslant 2\}$, $\psi(\xi) > 0$ for $\frac{1}{2} < |\xi| < 2$, and $\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) = 1$ for

$|\xi| > 0$. *This function* ψ *will be kept fixed in the rest of this note.* For $j = 0, \pm 1, \pm 2, \ldots$, we let ψ_j denote the function in $\mathscr{S}$ with $\hat{\psi}_j(\xi) = \mathscr{F}\psi_j(\xi) = \psi(2^{-j}\xi)$; $\mathscr{F}$ stands for the Fourier transform. We further let $\hat{\Psi}(\xi) = 1 - \sum_{j=1}^{\infty} \hat{\psi}_j(\xi)$. We define (cf. [5])

$$B_{p,q}^{s,w} = \left\{ f \in \mathscr{S}'; \|f\|_{B(s,w;p,q)} = \|\Psi * f\|_{p,w} + \left(\sum_{j=1}^{\infty} (2^{js} \|\psi_j * f\|_{p,w})^q \right)^{1/q} < \infty \right\},$$

$$\dot{B}_{p,q}^{s,w} = \left\{ f \in \mathscr{S}'; \|f\|_{\dot{B}(s,w;p,q)} = \left(\sum_{j=-\infty}^{\infty} (2^{js} \|\psi_j * f\|_{p,w})^q \right)^{1/q} < \infty \right\},$$

$$F_{p,q}^{s,w} = \left\{ f \in \mathscr{S}'; \|f\|_{F(s,w;p,q)} = \|\Psi * f\|_{p,w} + \left\| \left(\sum_{j=1}^{\infty} (2^{js} |\psi_j * f(\cdot)|)^q \right)^{1/q} \right\|_{p,w} < \infty \right\},$$

$$\dot{F}_{p,q}^{s,w} = \left\{ f \in \mathscr{S}'; \|f\|_{\dot{F}(s,w;p,q)} = \left\| \left(\sum_{j=-\infty}^{\infty} (2^{js} |\psi_j * f(\cdot)|)^q \right)^{1/q} \right\|_{p,w} < \infty \right\},$$

where $(\sum_j |a_j|^q)^{1/q}$ is interpreted as $\sup_j |a_j|$ if $q = \infty$; when $w = 1$, we shall drop w from the corresponding symbols. We observe that each of the above spaces does not depend on the choice of ψ by [5, Theorem 2.2]; in fact, another choice of the sequence $\{\psi_j\}$ and Ψ satisfying rather flexible conditions will define the same spaces with equivalent quasi-norms (cf. [5, 13, 14, 19]). We also notice that, when dealing with homogeneous spaces (spaces denoted with a dot), we have followed the usual practice of making calculus modulo polynomials.

The two scales of spaces defined above make their appearance in many branches of analysis, and we mention some examples here. The spaces $B_{p,q}^{s,w}$ are useful in describing weighted approximations by entire functions of exponential type; this approach has been taken by Löfström [12]. Moreover, it is not difficult to see that appropriate versions of approximation theorems in [19, 2.2.3] can be proved for $B_{p,q}^{s,w}$ and $F_{p,q}^{s,w}$. Second, we notice that regularity theorems for elliptic partial differential equations can be given in terms of $B_{p,q}^{s,w}$ and $F_{p,q}^{s,w}$ (cf. [5, 3.4]). Last, we observe the following two interesting cases of weighted Triebel–Lizorkin spaces (cf. [5, Sect. 4]):

$$\dot{F}_{p,2}^{0,w} = H_w^p \qquad \text{(mod. polynomials)},$$
$$F_{p,2}^{0,w} = h_w^p,$$

where H_w^p and h_w^p are weighted Hardy spaces whose definitions will be recalled in subsections 1.3 and 1.2.

Hereafter, the letter C or c will denote a positive constant whose value might change from one occurrence to the next one; we occasionally use subscripts to emphasize their dependence on the concerned parameters.

1.2. *Results for Nonhomogeneous Spaces*

We let $W_t(x) = W(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ denote the Gauss–Weierstrass kernel for R_+^{n+1}; note that $\hat{W}_t(\xi) = \exp(-4\pi^2 t\, |\xi|^2)$. Let f be a tempered distribution, u be its Gauss–Weierstrass integral (on R_+^{n+1}), i.e., $u(x, t) = W_t * f(x)$, and $v(x, t) = (\partial/\partial t)^k u(x, t)$, where k is a nonnegative integer greater than $s/2$; *f, u, v, and k will have the above meaning throughout this note unless otherwise stated.*

THEOREM 1. *Let $0 < \delta \leqslant \infty$ and $0 < \alpha, \beta < \gamma < \infty$. Then the following two propositions hold*:

(i) *If $f \in B_{p,q}^{s,w}$, then*

$$\mathscr{B}_{p,q}^{s,w}(f) = \left(\int_\beta^\gamma \|u(\cdot, t)\|_{h(p,w)}^\delta\, t^{-1}\, dt\right)^{1/\delta} + \left(\int_0^\alpha (t^{k-s/2}\, \|v(\cdot, t)\|_{h(p,w)})^q\, t^{-1}\, dt\right)^{1/q} \leqslant C\, \|f\|_{B(s,w;p,q)}$$

(*with the usual interpretation if $\delta = \infty$ or $q = \infty$*).

(ii) *Conversely, if*

$$B_{p,q}^{s,w}(f) = \left(\int_\beta^\gamma \|u(\cdot, t)\|_{p,w}^\delta t^{-1}\, dt\right)^{1/\delta} + \left(\int_0^\alpha (t^{k-s/2}\, \|v(\cdot, t)\|_{p,w})^q t^{-1}\, dt\right)^{1/q} < \infty,$$

then $f \in B_{p,q}^{s,w}$, and

$$\|f\|_{B(s,w;p,q)} \leqslant C B_{p,q}^{s,w}(f).$$

Here $\|g\|_{h(p,w)} = \|\sup_{0<t<1} |\Psi_t * g(\cdot)|\|_{p,w}$ for $g \in \mathcal{S}'$ is the quasi-norm on the nonhomogeneous weighted Hardy space h_w^p (cf. [5, Sect. 1]), where $\Psi_t(x) = t^{-n}\Psi(x/t)$. We derive from a well known property of the mollifier Ψ, that $\|g\|_{p,w} \leqslant \|g\|_{h(p,w)}$ for any function g such that

$$\int |g(x)|(1+|x|)^{-N}\,dx < \infty \qquad \text{for some } N;$$

we call such a function g a *function of temperate growth.* Since each $u(\cdot, t)$ or each $v(\cdot, t)$ is a function of temperate growth, we conclude that $B_{p,q}^{s,w}(f) \leqslant \mathscr{B}_{p,q}^{s,w}(f)$, and hence we obtain the following equivalence of quasi-norms:

$$\|f\|_{B(s,w;p,q)} \approx \mathscr{B}_{p,q}^{s,w}(f) \approx B_{p,q}^{s,w}(f)$$

for an arbitrary tempered distribution f.

COROLLARY 2. *Assume that either*

$$s > 0,\ 1 \leqslant p < \infty \qquad \textit{and} \qquad w \in A_p, \tag{1}$$

or

$$w \in \mathscr{M}_d\ (d > 0), \quad w \notin A_{r_0}, \quad 0 < p \leqslant r_0 \quad \textit{and} \quad s > d(1/p - 1/r_0), \tag{2}$$

or

$$w \in A_1 \cap \mathscr{M}_d\ (d > 0), \quad 0 < p < 1 \quad \textit{and} \quad s > d(1/p - 1). \tag{3}$$

Then $f \in B_{p,q}^{s,w}$ if and only if f is a function of temperate growth and

$$\|f\|_{p,w} + \left(\int_0^\alpha (t^{k-s/2}\,\|v(\cdot, t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q} \tag{4}$$

is finite. Furthermore, (4) *is an equivalent quasi-norm in $B_{p,q}^{s,w}$. (The weight function w is said to be in $\mathscr{M}_d$ if $\int_{\{|x-y|<\rho\}} w(y)\,dy \geqslant c\rho^d$ for $x \in R^n$ and $0 < \rho \leqslant 1$.)*

Remark 3. If either $s > nr_0/p$, or $w = 1$ and $s > \max(0, n(1/p - 1))$, then we can replace the second term of (4) by

$$\left(\int_0^\alpha (t^{k-s/2}\,\|v(\cdot, t)\|_{h(p,w)})^q t^{-1}\,dt\right)^{1/q}$$
$$+ \left(\int_\alpha^\infty (t^{k-s/2}\,\|v_\lambda^*(\cdot, t)\|_{p,w})^q t^{-1}\,dt\right)^{1/q}, \qquad \lambda > nr_0/p.$$

in the necessary part of Corollary 2. Here v_λ^* is defined by

$$v_\lambda^*(x,t) = \sup_y \{|v(x-y,t)|\,(1+|y|/\sqrt{t})^{-\lambda}\}.$$

The function $t^{-k}v_\lambda^*(\cdot,t)$ is a particular case of a maximal function used by Triebel [20] in deriving equivalent quasi-norms for nonweighted spaces, and the latter is the continuous version of the one introduced by Peetre [13] in his study of Triebel–Lizorkin spaces; all these maximal functions have their origin in the theory of Hardy spaces [7]. One of the purposes of this remark and Remark 6 is to relate our results with those of Triebel (see Section 2(A)). We also notice from a well known fact in the theory of weighted Hardy spaces that

$$\left(\int_a^b (t^{k-s/2}\,\|v_\lambda^*(\cdot,t)\|_{p,w})^q\,t^{-1}\,dt\right)^{1/q} \leqslant C\left(\int_a^b (t^{k-s/2}\,\|v(\cdot,t)\|_{h(p,w)})^q\,t^{-1}\,dt\right)^{1/q},$$

where $0 \leqslant a < b \leqslant \infty$. In fact, the above inequality can be proved by dividing the domain of the integral on the left and using [5, (2)].

THEOREM 4. *Let δ, α, β, and γ be as in Theorem* 1. *Assume $\lambda > \max(nr_0/p, n/q)$. Then the following two propositions hold*:

(i) *If $f \in F_{p,q}^{s,w}$, then*

$$\mathscr{F}_{p,q}^{s,w}(f) = \left(\int_\beta^\gamma \|u(\cdot,t)\|_{h(p,w)}^\delta\,t^{-1}\,dt\right)^{1/\delta} + \left\|\left(\int_0^\alpha (t^{k-s/2}v_\lambda^*(\cdot,t))^q\,t^{-1}\,dt\right)^{1/q}\right\|_{p,w} \leqslant C\,\|f\|_{F(s,w;p,q)}.$$

(ii) *Conversely, if*

$$F_{p,q}^{s,w}(f) = \left(\int_\beta^\gamma \|u(\cdot,t)\|_{p,w}^\delta\,t^{-1}\,dt\right)^{1/\delta} + \left\|\left(\int_0^\alpha (t^{k-s/2}\,|v(\cdot,t)|)^q\,t^{-1}\,dt\right)^{1/q}\right\|_{p,w} < \infty,$$

then $f \in F_{p,q}^{s,w}$, and

$$\|f\|_{F(s,w;p,q)} \leqslant CF_{p,q}^{s,w}(f).$$

Consequently,

$$\|f\|_{F(s,w;p,q)} \approx \mathscr{F}^{s,w}_{p,q}(f) \approx F^{s,w}_{p,q}(f)$$

for an arbitrary tempered distribution f.

COROLLARY 5. *Assume that either* (1), (2), *or* (3) *holds. Then* $f \in F^{s,w}_{p,q}$ *if and only if f is a function of temperate growth and*

$$\|f\|_{p,w} + \left\| \left(\int_0^{\infty} (t^{k-s/2} |v(\cdot,t)|)^q t^{-1}\, dt \right)^{1/q} \right\|_{p,w} \tag{5}$$

is finite. Furthermore, (5) *is an equivalent quasi-norm in* $F^{s,w}_{p,q}$.

Remark 6. In the necessary part of Corollary 5, if $s > \max(nr_0/p, n/q)$, then we can replace the second term in (5) by

$$\left\| \left(\int_0^{\infty} (t^{k-s/2} v^*_\lambda(\cdot,t))^q t^{-1}\, dt \right)^{1/q} \right\|_{p,w}, \qquad \lambda > \max(nr_0/p, n/q).$$

PROPOSITION 7. *Let* $0 < \mu < T < \infty$ *and let u be a temperature* (*i.e., a solution of the heat equation* $\Delta_x - (\partial/\partial t) = 0$) *on* $D_T = R^n \times]0, T[$. *Then the following two propositions hold*:

(i) *u is the Gauss–Weierstrass integral on* D_T *of a distribution f in* $B^{s,w}_{p,q}$ *if and only if*

$$\sum_{j=0}^{k-1} \|(\partial/\partial t)^j u(\cdot,\mu)\|_{p,w} + \left(\int_0^{T} (t^{k-s/2} \|(\partial/\partial t)^k u(\cdot,t)\|_{p,w})^q t^{-1}\, dt \right)^{1/q} \tag{6}$$

is finite for a nonnegative integer k greater than s/2.

(ii) *u is the Gauss–Weierstrass integral on* D_T *of a distribution g in* $F^{s,w}_{p,q}$ *if and only if*

$$\sum_{j=0}^{k-1} \|(\partial/\partial t)^j u(\cdot,\mu)\|_{p,w} + \left\| \left(\int_0^{T} (t^{k-s/2} |(\partial/\partial t)^k u(\cdot,t)|)^q t^{-1}\, dt \right)^{1/q} \right\|_{p,w} \tag{7}$$

is finite for a nonnegative integer k greater than s/2.

Furthermore, (6) (*resp.* (7)) *for a fixed k is equivalent to* $\|f\|_{B(s,w;p,q)}$ (*resp.* $\|g\|_{F(s,w;p,q)}$).

Remark 8. The first term in (6) or (7) can be replaced by

$$\left(\int_\beta^{\gamma} \|u(\cdot,t)\|^\delta_{p,w} t^{-1}\, dt \right)^{1/\delta}, \qquad 0 < \beta < \gamma < T, 0 < \delta \leqslant \infty,$$

and it can be dropped if $k=0$. We note that the main difference between Proposition 7 and Theorem 1 or Theorem 4 is that u is not assumed to be the Gauss–Weierstrass integral of a distribution in the former. We also observe that, even in the case $w=1$, the results of Proposition 7 seem known before only for $B^s_{p,q}$, $s<0=k$, $1\leqslant p, q\leqslant\infty$ (cf. [10, Corollary 3]).

1.3. *Results for Homogeneous Spaces*

THEOREM 1′. (i) *Assume that* $w\in\dot{\mathscr{M}}_d$, $d>0$, *i.e.*, $\int_{\{|x-y|<\rho\}} w(y)\,dy \geqslant c\rho^d$ *for* $x\in R^n$ *and* $0<\rho<\infty$. *Then for any* $f\in\dot{B}^{s,w}_{p,q}$, *there exists a polynomial P such that*

$$\dot{\mathscr{B}}^{s,w}_{p,q}(f)=\left(\int_0^\infty (t^{k-s/2}\,\|v(\cdot,t)\|_{H(p,w)})^q t^{-1}\,dt\right)^{1/q}\leqslant C\,\|f\|_{\dot{B}(s,w;p,q)},\tag{8}$$

where $v(\cdot,t)=(\partial/\partial t)^k[W_t*(f-P)]$.

(ii) *Conversely, if* $f\in\mathscr{S}'$ *and* $v(\cdot,t)=(\partial/\partial t)^k(W_t*f)$ *satisfies*

$$\dot{B}^{s,w}_{p,q}(f)=\left(\int_0^\infty (t^{k-s/2}\,\|v(\cdot,t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q}<\infty,$$

then $f\in\dot{B}^{s,w}_{p,q}$, *and*

$$\|f\|_{\dot{B}(s,w;p,q)}\leqslant C\dot{B}^{s,w}_{p,q}(f).\tag{9}$$

Here $\|\cdot\|_{H(p,w)}$ is the quasi-norm on the weighted Hardy space H^p_w, i.e., $\|g\|_{H(p,w)}=\|\sup_{0<t<\infty}|\Psi_t*g(\cdot)|\|_{p,w}$ for $g\in\mathscr{S}'$ (cf. [5, 7, 17]). Notice then that $\|g\|_{h(p,w)}\leqslant\|g\|_{H(p,w)}$. We also note that, for $w_\rho(x)=|x|^\rho$, $0\leqslant\rho<\infty$, an easy computation shows that $w_\rho\in\dot{\mathscr{M}}_{n+\rho}$.

THEOREM 4′. (i) *Assume that* $w\in\dot{\mathscr{M}}_d$, $d>0$, *and* $\lambda>\max(nr_0/p, n/q)$. *Then for any* $f\in\dot{F}^{s,w}_{p,q}$, *there exists a polynomial P such that*

$$\dot{\mathscr{F}}^{s,w}_{p,q}(f)=\left\|\left(\int_0^\infty (t^{k-s/2}v^*_\lambda(\cdot,t))^q\, t^{-1}\,dt\right)^{1/q}\right\|_{p,w}\leqslant C\,\|f\|_{\dot{F}(s,w;p,q)},\tag{10}$$

where $v(\cdot,t)=(\partial/\partial t)^k[W_t*(f-P)]$.

(ii) *Conversely, if* $f\in\mathscr{S}'$ *and* $v(\cdot,t)=(\partial/\partial t)^k(W_t*f)$ *satisfies*

$$\dot{F}^{s,w}_{p,q}(f)=\left\|\left(\int_0^\infty (t^{k-s/2}\,|v(\cdot,t)|)^q\, t^{-1}\,dt\right)^{1/q}\right\|_{p,w}<\infty,$$

then $f\in\dot{F}^{s,w}_{p,q}$, *and*

$$\|f\|_{\dot{F}(s,w;p,q)}\leqslant C\dot{F}^{s,w}_{p,q}(f).\tag{11}$$

We notice that the identification of $\dot{F}^{0,w}_{p,2}$ (resp. $F^{0,w}_{p,2}$) with the weighted Hardy spaces H^p_w (resp. h^p_w) observed in 1.1 and Theorem 4′ (resp. Theorem 4) yield another characterization of weighted Hardy spaces.

2. Remarks and Further Results

(A) In a recent paper [20], Triebel gave equivalent quasi-norms for Besov and Triebel–Lizorkin spaces in the nonweighted case ($w = 1$) by means of a function in a rather wide class that contains both derivatives of the Poisson kernel and those of the Gauss–Weierstrass kernel. We confine ourselves here to the F-space case and note that a similar remark holds for B-spaces. A main result of his for F-spaces is the following (cf. [20, Theorems 1(ii) and 3(ii)]):

Let ϕ be a nonnegative infinitely differentiable function on $R^n \setminus \{0\}$ and $\lambda > \max(n/p, n/q)$. Assume that $\phi(x) > 0$ for $\frac{1}{4} < |x| < 4$, and

$$\sup_{x \neq 0, |\kappa| \leq L} (|x|^{-L} + |x|^L)\, |D^\kappa \phi(x)| < \infty$$

for a sufficiently large positive integer L. Then

$$c \left\| \left(\int_0^\infty (t^{-s} v_\lambda^*(\cdot, t))^q t^{-1}\, dt \right)^{1/q} \right\|_p \leq \|f\|_{\dot{F}(s;p,q)} \leq C \left\| \left(\int_0^\infty (t^{-s} |v(\cdot, t)|)^q t^{-1}\, dt \right)^{1/q} \right\|_p \tag{12}$$

for all $f \in \dot{F}^s_{p,q}$, and

$$c \left[\|f\|_p + \left\| \left(\int_0^\infty (t^{-s} v_\lambda^*(\cdot, t))^q t^{-1}\, dt \right)^{1/q} \right\|_p \right] \leq \|f\|_{F(s;p,q)} \leq C \left[\|f\|_p + \left\| \left(\int_0^\infty (t^{-s} |v(\cdot, t)|)^q t^{-1}\, dt \right)^{1/q} \right\|_p \right] \tag{13}$$

for all $f \in F^s_{p,q}$ and $s > \max(n/p, n/q)$.

Here $v(\cdot, t) = \mathscr{F}^{-1}(\phi(t\,\cdot)\,\mathscr{F}f)$ for smooth ϕ with compact support, and it is defined by a limiting process for general ϕ, and

$$v_\lambda^*(x, t) = \sup_y \{|v(x - y, t)|\, (1 + |y|/t)^{-\lambda}\}$$

(cf. [20, pp. 279–280]). As usual, in the homogeneous case, one makes calculus modulo polynomials; thus, in the definition of v for (12), we might replace f by $f - P$ with a suitable polynomial P (cf. 1.3 and 3.5). His results.

though applicable to a wide class of functions, seem not to imply a sufficient condition for a distribution to be in $\dot{F}^s_{p,q}$ or $F^s_{p,q}$. Actually, in his proof of the second inequality in (12) (resp. (13)), the finiteness of $\|f\|_{\dot{F}(s;p,q)}$ (resp. $\|f\|_{F(s;p,q)}$) is used. Hence, our sufficient results for F-spaces (Theorems 4(ii) and 4′(ii)) seem new even in the case $w = 1$. Note that ϕ corresponds to $(4\pi^2 |\xi|^2)^k \exp(-4\pi^2 |\xi|^2)$ in the Gauss–Weierstrass kernel case, and a direct application of Triebel's results gives equivalent quasi-norms for F-spaces if $k \geqslant |s| + 6 \max(n/p, n/q) + n + 4$ and λ is sufficiently close to $\max(n/p, n/q)$, while we only need $k > s/2$ in Theorems 4 and 4′. We also observe that, by modifying the argument used in the proof of Theorem 4(i) and Triebel's argument, we can extend some of his results [20, Theorems 1 and 3(i), (ii)] to the present weighted case. It would be of interest to see whether the finiteness of the last term in (12) (resp. (13)) would imply that $f \in \dot{F}^s_{p,q}$ (resp. $f \in F^s_{p,q}$) under some additional assumption on ϕ (for example, $\phi \in \mathscr{S}$).

(B) While the proofs of the necessary parts of our theorems are rather standard via interpolation theorems or maximal function techniques, those of the sufficient parts are based on a special property of temperatures, the "sub-mean-value property" (Lemma 9). Since harmonic functions possess a similar property (cf. [7, Lemma 2]), a harmonic function treatment of the spaces considered in Section 1 can be similarly given. However, there are some differences that we would like to point out. First, it seems not possible to study nonhomogeneous spaces of negative order by means of the Poisson kernel; even when $w = 1$, there is no way to define the Poisson integral of a distribution in $B^s_{p,q}$, $s < 0$. At least, if either (1), (2), or (3) is satisfied, then the harmonic function versions of Corollaries 2 and 5 hold, where we assume $k > s$ and, moreover $\int |f(x)|(1 + |x|)^{-n-1}\, dx < \infty$ in the sufficient parts. As for homogeneous spaces, the necessary parts of Theorems 1′ and 4′ hold if $k > s$ and W_t is replaced by P_t, the Poisson kernel for R^{n+1}_+; note that the Poisson integral of a distribution in $\dot{B}^{s,w}_{p,q}$ or $\dot{F}^{s,w}_{p,q}$ can be defined by a limiting process. On the other hand, since the Poisson integral of an arbitrary distribution cannot be defined, instead of the sufficient part of Theorem 1′ (resp. Theorem 4′), we turn to the study of a space of harmonic functions on R^{n+1}_+, which satisfy certain growth conditions at infinity ($t \to \infty$), equipped with a quasi-norm corresponding to $\dot{B}^{s,w}_{p,q}(\cdot)$ (resp. $\dot{F}^{s,w}_{p,q}(\cdot)$), and then we show that this space is isomorphic to $\dot{B}^{s,w}_{p,q}$ (resp. $\dot{F}^{s,w}_{p,q}$) by the operation of taking boundary values in $\mathscr{S}'/\mathscr{P}$, the space of distributions modulo polynomials. Details are not given here, but we should note that such a study in the case $w = 1$ has been done for homogeneous Besov spaces in [3] ($1 \leqslant p, q \leqslant \infty$) and [15] ($n = 1, 0 < p, q \leqslant \infty, s < 1/p$). In connection with this, we also notice that the characterizations of $\dot{F}^s_{p,q}$ ($1 < p, q < \infty$) via harmonic functions have been given by G. A. Kaljabin (cf. [20, Note added in proof]).

(C) Finally, we observe that our results might be useful in finding necessary and/or sufficient conditions for continuous convolution operators between weighted Besov and/or weighted Hardy spaces. Partial results in this direction are being obtained, and we hope to return to this subject at a later occasion (see [4, 11] for the case $w=1$).

3. Proofs

3.1. *Proof of Theorem* 1

We assume that $\alpha=1$, $\beta=1$, and $\gamma=2$ for the sake of simplicity of notations. We begin with the proof of (i). We first observe from a Fourier multiplier's criterion for weighted Hardy spaces [5, Lemma 4.8 or Theorem 2.4(ii)] that

$$\begin{aligned}\|(\partial/\partial t)^m(W_t * g)\|_{h(p,w)} &\leqslant C_m t^{-m}\|g\|_{h(p,w)} \qquad (14)\\ &\leqslant C_m t^{-m}\left(\|\Psi * g\|^{\rho}_{h(p,w)}+\sum_{j=1}^{\infty}\|\psi_j * g\|^{\rho}_{h(p,w)}\right)^{1/\rho}.\end{aligned}$$

$$\|J^{\mu}W_t * g\|_{h(p,w)} \leqslant C_{\mu}(1+t^{\mu/2})\|g\|_{h(p,w)} \tag{15}$$

for any nonnegative integer m, real number μ and $0<t\leqslant 1$. Here J^{μ} is the Bessel potential operator defined by $\mathscr{F}(J^{\mu}g)=(1+4\pi^2|\xi|^2)^{-\mu/2}\mathscr{F}g$ for $g\in\mathscr{S}'$, and $\rho=\min(1,p)$; note also that $g=\Psi * g+\sum_{j=1}^{\infty}\psi_j * g$ in $\mathscr{S}'$. The first inequality in (14) easily implies that

$$\begin{aligned}&\left(\int_0^1 (t^{k-s/2}\|v(\cdot,t)\|_{h(p,w)})^q t^{-1}\,dt\right)^{1/q}\\ &\qquad\approx\left(\sum_{j=0}^{\infty}(2^{j(s/2-k)}\|v(\cdot,2^{-j})\|_{h(p,w)})^q\right)^{1/q}. \qquad (16)\end{aligned}$$

Assume that $s<0$. Let s_0 and s_1 be negative numbers such that $s_0<s<s_1$. Then, it follows from (14) and (15) that

$$\begin{aligned}\|v(\cdot,2^{-j})\|_{h(p,w)} &= \|J^{s_i}W(\cdot,2^{-j-1}) * (\partial/\partial t)^k W(\cdot,2^{-j-1}) * J^{-s_i}f\|_{h(p,w)}\\ &\leqslant C2^{j(k-s_i/2)}\left(\|\Psi * J^{-s_i}f\|^{\rho}_{h(p,w)}+\sum_{m=1}^{\infty}\|\psi_m * J^{-s_i}f\|^{\rho}_{h(p,w)}\right)^{1/\rho}\\ &\leqslant C2^{j(k-s_i/2)}\|f\|_{B(s_i,w;p,\rho)}, \qquad (17)\end{aligned}$$

$j=0,1,2,\ldots$. We have used [5, Corollary 2.3 and Theorem 2.8(i)] in deriving the last inequality of (17). Consider now the linear map

$$T\colon f\mapsto\{v(\cdot,2^{-j})\}_{j=0}^{\infty}.$$

Then (17) shows that T continuously maps $B^{s_i,w}_{p,\min(1,p)}$ into $l^{s_i/2-k}_{\infty}(h^p_w)$, where $l^{\mu}_q(A)=\{\{a_j\};\ \|\{a_j\}\|=(\sum_j (2^{j\mu}\|a_j\|_A)^q)^{1/q}<\infty\}$ for a quasi-normed space A. Hence T continuously maps $B^{s,w}_{p,q}$ into $l^{s/2-k}_q(h^p_w)$ by interpolation theorem [5, Theorem 3.3(ii)], i.e.,

$$\left(\sum_{j=0}^{\infty} (2^{j(s/2-k)}\|v(\cdot,2^{-j})\|_{h(p,w)})^q\right)^{1/q} \leqslant C\|f\|_{B(s,w;p,q)}. \tag{18}$$

On the other hand, we see again from (14) and (15) that

$$\begin{aligned}\sup_{1\leqslant t\leqslant 2}\|u(\cdot,t)\|_{h(p,w)} &\approx \|u(\cdot,1)\|_{h(p,w)} = \|J^{s_0}W(\cdot,1)*J^{-s_0}f\|_{h(p,w)}\\ &\leqslant C\|f\|_{B(s_0,w;p,p)} \leqslant C\|f\|_{B(s,w;p,q)}\end{aligned} \tag{19}$$

for $s_0<s$ (we need not assume $s<0$ in (19)). The proof of (i) in the case $s<0$ is thus completed by combining (16) and (18) with (19). Next assume that $s\geqslant 0$ and $s_0<s<s_1<2k$. Then the result just proved for the case $s<0$ implies that

$$\begin{aligned}2^{j(s_i/2-k)}\|v(\cdot,2^{-j})\|_{h(p,w)} &= 2^{j(s_i/2-k)}\|W(\cdot,2^{-j})*(-\Delta)^k f\|_{h(p,w)}\\ &\leqslant C\|\Delta^k f\|_{B(s_i-2k,w;p,\infty)} \leqslant C\|f\|_{B(s_i,w;p,\infty)},\end{aligned}$$

$j=0,1,2,\ldots$. The desired result for the case $s\geqslant 0$ then follows by using the interpolation theorem and (19) as above. The proof of (i) in all cases is thus complete.

Before proceeding on with the proof of (ii), we recall a "sub-mean-value property" of temperatures.

LEMMA 9 (cf. [4, Lemma 2]). *Let h be a temperature on R^{n+1}_+ and $(x,t)\in R^{n+1}_+$. Assume $\theta>0$, $0<\theta^2<t$, and $0<r<\infty$. Then the inequality*

$$|h(x,t)|^r \leqslant C_r\theta^{-n-2}\iint_{\{|x_j-y_j|\leqslant\theta/2,\,j=1,\ldots,n,\,t-\theta^2\leqslant\rho\leqslant t\}} |h(y,\rho)|^r\,dy\,d\rho$$

holds.

Assume that the assumptions in (ii) of Theorem 1 hold. Fix $j=1,2,\ldots$, and let $t_j=2^{-2j}$. We write

$$\psi_j * f = t_j^k\phi_j * v(\cdot,t_j),$$

where $\phi_j(x) = t_j^{-n/2}\Phi(x/\sqrt{t_j})$, and $\hat{\Phi}(\xi) = \psi(\xi)(-4\pi^2|\xi|^2)^{-k}\exp(4\pi^2|\xi|^2)$ is a function in $\mathscr{S}$. Then it follows that

$$\psi_j * f(x) = t_j^k \int \Phi(y)\, v(x - \sqrt{t_j}\, y, t_j)\, dy = t_j^k \sum_{\mu \in Z^n} \int_{I_\mu} \Phi(y)\, v(x - \sqrt{t_j}\, y, t_j)\, dy, \tag{20}$$

where I_μ is the unit cube with centre at μ. Take $0 < r < \min(p, q)$ such that $p/r > r_0$ (hence $w \in A_{p/r}$). Since

$$c_1|\mu| \leqslant |y| \leqslant c_2(1 + |\mu|) \qquad \text{for} \quad y \in I_\mu,$$

we derive from Lemma 9 that

$$\begin{aligned}
|v(x - \sqrt{t_j}\, y, t_j)|^r &\leqslant C t_j^{-1} \int_{t_j/2}^{t_j} \Big((1 + |\mu|)^n |\omega_n (c(1 + |\mu|)\sqrt{t_j})^n|^{-1} \\
&\qquad \times \int_{\{|z - x| \leqslant c(1 + |\mu|)\sqrt{t_j}\}} |v(z, t)|^r\, dz \Big)\, dt \\
&\leqslant C(1 + |\mu|)^n \int_{t_j/2}^{t_j} M(|v_t|^r)(x)\, t^{-1}\, dt, \qquad y \in I_\mu,
\end{aligned} \tag{21}$$

where $v_t = v(\cdot, t)$, ω_n is the (Lebesgue) measure of the unit ball in R^n, and Mg is the Hardy maximal function of a locally integrable function g. It follows from (20) and (21) that

$$\|\psi_j * f\|_{p,w} \leqslant C_\Phi \left[\int \left(\int_{t_j/2}^{t_j} t^{kr} M(|v_t|^r)(x)\, t^{-1}\, dt \right)^{p/r} w(x)\, dx \right]^{1/p},$$

$j = 1, 2, \ldots$, and thus, we obtain

$$\begin{aligned}
c &\left(\sum_{j=1}^{\infty} (2^{js} \|\psi_j * f\|_{p,w})^q \right)^{1/q} \\
&\leqslant \left\{ \sum_{j=1}^{\infty} \left[\int_{t_j/2}^{t_j} \left(\int M(|v_t|^r)(x)^{p/r} w(x)\, dx \right)^{r/p} t^{kr - sr/2} t^{-1}\, dt \right]^{q/r} \right\}^{1/q} \\
&\leqslant C \left\{ \sum_{j=1}^{\infty} \left[\int_{t_j/2}^{t_j} \left(\int |v(x, t)|^p\, w(x)\, dx \right)^{r/p} t^{kr - sr/2} t^{-1}\, dt \right]^{q/r} \right\}^{1/q} \\
&\leqslant C \left(\sum_{j=1}^{\infty} \int_{t_j/2}^{t_j} (t^{k - s/2} \|v(\cdot, t)\|_{p,w})^q t^{-1}\, dt \right)^{1/q} \\
&\leqslant C \left(\int_0^1 (t^{k - s/2} \|v(\cdot, t)\|_{p,w})^q t^{-1}\, dt \right)^{1/q}
\end{aligned}$$

by using Minkowski's inequality, Muckenhoupt's weighted estimate for the Hardy maximal function (cf. [5, Lemma 1.1] or [6, Theorem I]) and Hölder's inequality, respectively.

To estimate $\|\Psi * f\|_{p,w}$, we write $\Psi * f = \phi * u(\cdot, 2)$, where $\hat{\phi}(\xi) = \hat{\Psi}(\xi)\exp(8\pi^2 |\xi|^2)$ is a function in $\mathscr{S}$. Let $\rho = \min(r, \delta)$. Then, an argument based on Lemma 9 and similar to the above implies that

$$\|\Psi * f\|_{p,w} \leqslant C\left[\int\left(\int_1^2 M(|u_t|^\rho)(x)\, t^{-1}\, dt\right)^{p/\rho} w(x)\, dx\right]^{1/p}$$
$$\leqslant C\left(\int_1^2 \|u(\cdot, t)\|_{p,w}^\rho\, t^{-1}\, dt\right)^{1/\rho} \leqslant C\left(\int_1^2 \|u(\cdot, t)\|_{p,w}^\delta\, t^{-1}\, dt\right)^{1/\delta},$$

where we have used Hölder's inequality in deriving the last inequality. The proof of (ii) is thus complete.

We notice that the main difference with the proof given in [4] (and also [14]) for the case $w = 1$ is the introduction of the Hardy maximal function in the estimate (21), which allows us to overcome the nontranslation invariant character of the measure wdx.

3.2. *Proofs of Corollary* 2 *and Remark* 3

Let $\rho = \min(1, p)$. We first observe that if $s > 0$ and $f \in B_{p,\infty}^{s,w}$, then for any positive integer N,

$$\left\|\sum_{j=m}^{m+N} \psi_j * f\right\|_{p,w} \leqslant \left(\sum_{j=m}^{\infty} \|\psi_j * f\|_{p,w}^\rho\right)^{1/\rho}$$
$$\leqslant C\|f\|_{B(s,w;p,\infty)}\left(\sum_{j=m}^{\infty} 2^{-js\rho}\right)^{1/\rho} \to 0$$

as $m \to \infty$. Therefore, there exists a function $b(f)$ in L_w^p such that

$$\Psi * f + \sum_{j=1}^{m} \psi_j * f \to b(f) \qquad \text{in } L_w^p \tag{22}$$

as $m \to \infty$, and

$$\|b(f)\|_{p,w} \leqslant C\|f\|_{B(s,w;p,\infty)}.$$

Assume now that (1) holds. Since $L_w^p \subset \mathscr{S}'$ (continuous embedding) in this case, and since the left-hand side of (22) always converges to f in $\mathscr{S}'$, we conclude that $b(f) = f$ and

$$\|f\|_{p,w} \leqslant C\|f\|_{B(s,w;p,\infty)}. \tag{23}$$

Assume next that either (2) or (3) holds. Take $r_0 < r < \infty$ such that $s > d(1/p - 1/r)$ in case of (2), and take $r = 1$ in case of (3). Then the embedding theorem [5, Theorem 2.6(iv)] implies that $f \in B_{r,\infty}^{s-d(1/p-1/r)}$. Since $s - d(1/p - 1/r) > 0$ and $w \in A_r$, assumption (1) holds with $p = r$, and hence, by what has been just proved above, we obtain the convergence

$$\Psi * f + \sum_{j=1}^{m} \psi_j * f \to f \text{ in } L_w^r,$$

which, together with (22), implies that $f = b(f)$ and (23) holds. Since $B_{p,q}^{s,w}$ is continuously embedded in $B_{p,\infty}^{s,w}$, we have proved that, if either (1), (2), or (3) holds, and $f \in B_{p,q}^{s,w}$, then f is a function of temperate growth and

$$\|f\|_{p,w} \leqslant C \|f\|_{B(s,w;p,q)}.$$

This observation and Theorem 1(i) imply the necessary part of Corollary 2.

For the sufficiency, we first note that

$$\begin{aligned} \|\Psi * f\|_{p,w} &\leqslant \left(\|f\|_{p,w}^{\rho} + \sum_{j=1}^{\infty} \|\psi_j * f\|_{p,w}^{\rho}\right)^{1/\rho} \\ &\leqslant C \left[\|f\|_{p,w}^{\rho} + \left(\sup_{1 \leqslant j \leqslant \infty} 2^{js} \|\psi_j * f\|_{p,w}\right)^{\rho}\right]^{1/\rho}. \end{aligned}$$

Next, we observe from the proof of Theorem 1(ii) that

$$\left(\sum_{j=1}^{\infty} (2^{js} \|\psi_j * f\|_{p,w})^q\right)^{1/q} \leqslant C \left(\int_0^{\alpha} (t^{k-s/2} \|v(\cdot, t)\|_{p,w})^q t^{-1}\, dt\right)^{1/q}.$$

These two estimates imply the sufficient part of the corollary and also the equivalence of quasi-norms.

Finally, we turn to the proof of Remark 3. Let $f \in B_{p,q}^{s,w}$ and $g = f - \Psi * f$. Then $g \in \dot{B}_{p,q}^{s,w}$, and $\hat{g} = 0$ in a neighborhood of the origin. The proof outlined in the homogeneous case (see 3.5) implies that

$$\begin{aligned} \left(\int_0^{\infty} (t^{k-s/2} \|(\partial/\partial t)^k W_t * g\|_{H(p,w)})^q t^{-1}\, dt\right)^{1/q} &\leqslant C \|g\|_{\dot{B}(s,w;p,q)} \\ &\leqslant C \|f\|_{B(s,w;p,q)}. \end{aligned}$$

Let $s > \lambda > nr_0/p$, $\hat{\Phi}(\xi) = (-4\pi^2 |\xi|^2)^k \exp(-4\pi^2 |\xi|^2)$ and $V(x, t) = (\partial/\partial t)^k [W_t * (g - f)]$. Since

$$|V(x-y,t)| \leqslant t^{-k}\int |\Phi(z)|(1+|y|+\sqrt{t}\,|z|)^{\lambda}$$
$$\times |\Psi * f(x-y-\sqrt{t}z)|(1+|y+\sqrt{t}z|)^{-\lambda}\,dz$$
$$\leqslant C_{\Phi,\alpha}t^{-k+\lambda/2}(1+|y|/\sqrt{t})^{\lambda}$$
$$\times [\sup_{y}\{|\Psi * f(x-y)|(1+|y|)^{-\lambda}], \qquad t \geqslant \alpha,$$

we conclude that

$$\left(\int_{\alpha}^{\infty}(t^{k-s/2}\,\|V_{\lambda}^{*}(\cdot,t)\|_{p,w})^{q}t^{-1}\,dt\right)^{1/q} \leqslant C\,\|\Psi * f\|_{p,w} \leqslant C\,\|f\|_{B(s,w;p,q)}$$

by [5, Lemma 2.1]. The proof of Remark 3 in the case $s > nr_0/p$ is thus complete.

Next assume $w = 1$ and $s > \max(0, n(1/p-1))$. Since the desired result is classical and well known for $p > 1$, we consider only the case $p \leqslant 1$. For any $t \geqslant \alpha$,

$$\|v(\cdot,t)\|_{h^p} \leqslant C\,\|((\partial/\partial t)^k W_t) * f\|_{B(0;p,p)}$$
$$\leqslant C\,\|(\partial/\partial t)^k W_t\|_{B(n(1/p-1);p,\infty)}\,\|f\|_{B(0;p,p)}$$
$$\leqslant Ct^{-k+n(1/p-1)/2}\,\|f\|_{B(s;p,q)}$$

by the relation $h^p = F^0_{p,2}$ [5, Sect. 4], the embedding $B^0_{p,p} \subset F^0_{p,2}$ [5, Theorem 2.6(ii)], and a convolution theorem on Besov spaces [14, Chap. 11, Theorem 14; 18, II., Lemma 1], and the estimate

$$\|(\partial/\partial t)^k W_t\|_{B(n(1/p-1);p,\infty)} \leqslant C_{\alpha}t^{-k+n(1/p-1)/2}, \qquad t \geqslant \alpha.$$

The last inequality is easily proved by using Theorem 1(ii) and the formula $(\partial/\partial t)^k W(x,t) = t^{-k}W(x,t)\,Q(|x|^2/4t)$ for a polynomial Q of degree k. The proof of the other assertion of Remark 3 is thus complete. Notice that we have actually proved that $\|v_{\lambda}^{*}(\cdot,t)\|_p$ can be replaced by $\|v(\cdot,t)\|_{h^p}$ in the whole range $0 < t < \infty$.

3.3. *Proof of Theorem* 4

Again we assume $\alpha = 1$, $\beta = 1$, and $\gamma = 2$ for the sake of simplicity of notations. We begin with the proof of (ii). We retain the notations used in the proof of Theorem 1(ii). For each $j = 1, \ldots,$ we put

$$g_j(z) = \int_{t_j/2}^{t_j} |v(z,t)|^r t^{-1}\,dt.$$

Then, the first inequality of (21) gives

$$|v(x-\sqrt{t_j}\,y, t_j)|^r \leqslant C(1+|\mu|)^n Mg_j(x), \qquad y \in I_\mu,$$

which, together with (20), implies that

$$\left\| \left(\sum_{j=1}^{\infty} (2^{js} |\psi_j * f(x)|)^q \right)^{1/q} \right\|_{p,w}$$

$$\leqslant C \left[\int \left(\sum_{j=1}^{\infty} (M(2^{-2jkr+jsr} g_j)(x))^{q/r} \right)^{p/q} w(x)\,dx \right]^{1/p} = CI(f).$$

Therefore, it follows from the weighted estimate for the vector-valued Hardy maximal function (cf. [5, Lemma 1.1]) and Hölder's inequality that

$$I(f) \leqslant C \left[\int \left(\sum_{j=1}^{\infty} 2^{j(s-2k)q} |g_j(x)|^{q/r} \right)^{p/q} w(x)\,dx \right]^{1/p}$$

$$= C \left\{ \int \left[\sum_{j=1}^{\infty} 2^{j(s-2k)q} \left(\int_{t_j/2}^{t_j} |v(x,t)|^r t^{-1}\,dt \right)^{q/r} \right]^{p/q} w(x)\,dx \right\}^{1/p}$$

$$\leqslant C \left[\int \left(\sum_{j=1}^{\infty} \int_{t_j/2}^{t_j} (t^{k-s/2} |v(x,t)|)^q\, t^{-1}\,dt \right)^{p/q} w(x)\,dx \right]^{1/p}$$

$$\leqslant C \left\| \left(\int_0^1 (t^{k-s/2} |v(x,t)|)^q\, t^{-1}\,dt \right)^{1/q} \right\|_{p,w}, \qquad q < \infty.$$

We obtain similar estimates in the case $q = \infty$ by using the obvious inequality

$$\sup_j M(2^{-2jkr+jsr} g_j) \leqslant M(\sup_j (2^{-2jkr+jsr} g_j))$$

and the weighted estimate for the Hardy maximal function instead of its vector-valued version. Since $F^{s,w}_{p,q} \subset B^{s,w}_{p,\infty}$ [5, Theorem 2.6(ii), (iii)], the estimate for $\|\Psi * f\|_{p,w}$ follows from Theorem 1(ii). The proof of (ii) is thus complete.

We now turn to the proof of (i). The estimate for the first terms of $\mathscr{F}^{s,w}_{p,q}(f)$ is deduced from Theorem 1(i) via the embedding $F^{s,w}_{p,q} \subset B^{s,w}_{p,\infty}$. For the second term, we note for each x and t,

$$|v(x,t)| \leqslant |\Psi * W_t * (-\Delta)^k f(x)| + t^{-k} \left| \sum_{j=1}^{\infty} \Phi_{\sqrt{t}} * \psi_j * \phi_j * f(x) \right|$$

$$= |U(x,t)| + |V(x,t)|,$$

where $\hat{\Phi}(\xi) = (-4\pi^2 |\xi|^2)^k \exp(-4\pi^2 |\xi|^2)$, and $\hat{\phi}_j(\xi) = \phi(2^{-j}\xi)$, $j = 0, \pm 1,$

$\pm 2, \ldots$, for some $\phi \in \mathscr{S}$ which satisfies: $\operatorname{supp} \phi \subset \{\frac{1}{3} \leqslant |\xi| \leqslant 3\}$, $\phi(\xi) = 1$ for $\frac{1}{2} \leqslant |\xi| \leqslant 2$. It is easily seen that

$$\begin{aligned} &|U(x-y, t)| \\ &\quad \leqslant C(1 + |y|/\sqrt{t})^{\lambda} [\sup_z \{|(\Psi * \Delta^k f)(x - y - \sqrt{t} z)|(1 + |y + \sqrt{t} z|)^{-\lambda}\}], \\ &\qquad\qquad t \leqslant 1. \end{aligned}$$

Since $\Delta^k f \in F_{p,q}^{s-2k,w}$, we derive from the above inequality and [5, Lemma 2.1] that

$$\begin{aligned} \left\| \left(\int_0^1 (t^{k-s/2} U_\lambda^*(x,t))^q t^{-1}\, dt \right)^{1/q} \right\|_{p,w} &\leqslant C \| \Psi * \Delta^k f \|_{p,w} \\ &\leqslant C \|\Delta^k f\|_{F(s-2k, w; p,q)} \leqslant C \|f\|_{F(s,w;p,q)}. \end{aligned} \tag{24}$$

To estimate V_λ^*, we use the familiar technique of maximal function (cf. [13, pp. 126–127]). For each $\nu = 0, 1, 2, \ldots$, and $2^{-\nu-1} \leqslant t \leqslant 2^{-\nu}$, we have

$$\begin{aligned} &ct^{2k-s} |V(x-y, t^2)| \\ &\quad \leqslant \sum_{j=1}^{\infty} 2^{(\nu-j)s} (1 + 2^{(j-\nu)\lambda}) \left(\int 2^{\nu n} |\Phi_{2^\nu t} * \psi_{j-\nu}(2^\nu z)|(1 + 2^j |z|)^{\lambda}\, dz \right) \\ &\qquad \times 2^{js} [\sup_z \{|\phi_j * f(x - y - z)|(1 + 2^j |y + z|)^{-\lambda}\}](1 + |y|/t)^{\lambda}. \end{aligned}$$

Thus,

$$t^{2k-s} V_\lambda^*(x, t^2) \leqslant C \sum_{j=1}^{\infty} a_{j-\nu} (2^{(\nu-j)s} + 2^{(\nu-j)(s-\lambda)})(2^{js} \phi_{j\lambda}^* f(x)),$$

where $\phi_{j\lambda}^*$ denotes the term inside the brackets $[\ \cdots\ ]$, and

$$a_j = \sup_{1/2 \leqslant \mu \leqslant 1} \int |\Phi_\mu * \psi_j(z)|\, (1 + 2^j |z|)^{\lambda}\, dz, \qquad j = 0, \pm 1, \pm 2, \ldots .$$

Hence, it follows that

$$\begin{aligned} &\left(\int_0^1 (t^{k-s/2} V_\lambda^*(x,t))^q t^{-1}\, dt \right)^{1/q} \\ &\quad = \left(\sum_{\nu=0}^{\infty} 2 \int_{2^{-\nu-1}}^{2^{-\nu}} (t^{2k-s} V_\lambda^*(x, t^2))^q\, t^{-1}\, dt \right)^{1/q} \\ &\quad \leqslant C \left(\sum_{j=-\infty}^{\infty} [a_j (2^{-js} + 2^{-j(s-\lambda)})]^{\rho} \right)^{1/\rho} \left(\sum_{j=1}^{\infty} (2^{js} \phi_{j\lambda}^* f(x))^q \right)^{1/q} \\ &\quad = C S_k \left(\sum_{j=1}^{\infty} (2^{js} \phi_{j\lambda}^* f(x))^q \right)^{1/q}, \qquad \rho = \min(1, q). \end{aligned}$$

Consequently, we obtain

$$\left\| \left(\int_0^1 (t^{k-s/2} V_\lambda^*(x,t))^q \, t^{-1} \, dt \right)^{1/q} \right\|_{p,w} \leqslant CS_k \|f\|_{F(s,w;p,q)} \tag{25}$$

by [5, Theorem 2.2]. Thus, the desired result follows from (24) and (25) provided that $S_k < \infty$ for any $k > s/2$. To prove this last assertion, take a positive integer $N > \lambda + n/2$. Then, we see from Schwarz's inequality that

$$a_{j\mu} = \int (1 + 2^j |z|)^\lambda \, |\Phi_\mu * \psi_j(z)| \, dz \leqslant C2^{-jn/2} \|(1 + 2^j |z|)^N (\Phi_\mu * \psi_j)\|_2. \tag{26}$$

We consider the case $j \leqslant 0$ first. Plancherel's theorem, and the behaviors of $\hat{\Phi}$ and its derivatives at the origin imply that

$$\begin{aligned} \|\Phi_\mu * \psi_j\|_2 &= \left(\int_{\{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}} |\psi(2^{-j}\xi)|^2 \, |\hat{\Phi}(\mu\xi)|^2 \, d\xi \right)^{1/2} \\ &\leqslant C2^{j(2k+n/2)}, \end{aligned}$$

and

$$\begin{aligned} \|z_i^N(\Phi_\mu * \psi_j)\|_2 &= c \left(\int_{\{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}} |(\partial/\partial\xi_i)^N [\psi(2^{-j}\xi) \, \hat{\Phi}(\mu\xi)]|^2 \, d\xi \right)^{1/2} \\ &\leqslant C2^{j(2k-N+n/2)}, \qquad i = 1, 2, \ldots, n, \end{aligned}$$

where the constants C appearing above are independent of μ, $\frac{1}{2} \leqslant \mu \leqslant 1$. Since $(1 + 2^j |z|)^N \leqslant C(1 + \sum_{i=1}^n 2^{jN} |z_i^N|)$, we derive from (26) and the above estimates that

$$a_j = \sup_{1/2 \leqslant \mu \leqslant 1} a_{j\mu} \leqslant C2^{2kj}, \qquad j \leqslant 0,$$

which implies

$$\begin{aligned} &\left(\sum_{j=-\infty}^{0} |a_j(2^{-js} + 2^{-j(s-\lambda)}|^p \right)^{1/p} \\ &\qquad \leqslant C \left(\sum_{j=-\infty}^{0} |2^{j(2k-s)p} + 2^{j(2k-s-\lambda)p}| \right)^{1/p} < \infty \end{aligned}$$

if $k > s/2$. The summation over $j > 0$ can be handled in a similar way, but in this case we must take into consideration the behaviors of $\hat{\Phi}$ and its derivatives at infinity (instead of their behaviors at the origin); since $\hat{\Phi}(\xi) = (-4\pi^2 |\xi|^2)^k \exp(-4\pi^2 |\xi|^2)$, no condition on k is needed in this case. The proof of the assertion on S_k is thus complete, and so is the proof of Theorem 4.

Corollary 5 and Remark 6 then follow from the embedding $F_{p,q}^{s,w} \subset B_{p,r}^{s,w}$ and an argument similar to the proofs of Corollary 2 and Remark 3.

3.4. *Proof of Proposition* 7

We first give another corollary of Theorems 1 and 4.

COROLLARY 10. *Let α and β be positive numbers, and let f, u, v, and k have the same meaning as at the beginning of subsection* 2.1. *Then the following two propositions hold*:

(i) $f \in B_{p,q}^{s,w}$ *if and only if*

$$\sum_{j=0}^{k-1} \|(\partial/\partial t)^j u(\cdot,\beta)\|_{p,w} + \left(\int_0^\alpha (t^{k-s/2}\,\|v(\cdot,t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q} \tag{27}$$

is finite.

(ii) $f \in F_{p,q}^{s,w}$ *if and only if*

$$\sum_{j=0}^{k-1} \|(\partial/\partial t)^j u(\cdot,\beta)\|_{p,w} + \left\|\left(\int_0^\alpha (t^{k-s/2}\,|v(\cdot,t)|)^q\, t^{-1}\,dt\right)^{1/q}\right\|_{p,w} \tag{28}$$

is finite.

Furthermore, (27) (*resp.* (28)) *is an equivalent quasi-norm on $B_{p,q}^{s,w}$* (*resp.* $F_{p,q}^{s,w}$). (*Note also that the first term in* (27) *or* (28) *can be dropped if $k=0$.*)

Proof. Since $(\partial/\partial t)^j u(\cdot,\mu) = W_\mu * (-\Delta)^j f$, the necessary parts follow from Theorems 1(i) and 4(i). We prove only the sufficient part for (i), because that for (ii) can be similarly treated. We assume that $\alpha = 1$ and $\beta = 2$ for the sake of simplicity of notations. Noting that $v(\cdot,t) = W_t * (-\Delta)^k f$, we derive from Theorem 1 that $(-\Delta)^k f \in B_{p,q}^{s-2k,w}$ and

$$\left(\int_0^2 (t^{k-s/2}\,\|v(\cdot,t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q} \leqslant C\left(\int_0^1 (t^{k-s/2}\,\|v(\cdot,t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q}.$$

The above inequality, the formula

$$(\partial/\partial t)^{k-1} u(x,\mu) = -\int_\mu^2 v(x,t)\,dt + (\partial/\partial t)^{k-1} u(\mu,2), \qquad \tfrac{1}{2} \leqslant \mu < 2,$$

and an argument (based on Lemma 9 and) similar to the proof of Theorem 1(ii) imply that

$$\begin{aligned}
&\sup_{1/2 \leqslant \mu \leqslant 2} \|(\partial/\partial t)^{k-1} u(\cdot,\mu)\|_{p,w} \\
&\quad\leqslant C\left[\left(\int_0^2 (t^{k-s/2}\,\|v(\cdot,t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q} + \|(\partial/\partial t)^{k-1} u(\cdot,2)\|_{p,w}\right] \\
&\quad\leqslant C\left[\left(\int_0^1 (t^{k-s/2}\,\|v(\cdot,t)\|_{p,w})^q\, t^{-1}\,dt\right)^{1/q} + \|(\partial/\partial t)^{k-1} u(\cdot,2)\|_{p,w}\right].
\end{aligned}$$

Repeating the above arguments if necessary, we obtain

$$\sup_{1 \leqslant t \leqslant 2} \|u(\cdot, t)\|_{p,w} \leqslant C \left[\left(\int_0^1 (t^{k-s/2} \|v(\cdot, t)\|_{p,w})^q t^{-1} \, dt \right)^{1/q} + \sum_{j=0}^{k-1} \|(\partial/\partial t)^j u(\cdot, 2)\|_{p,w} \right].$$

Combining these facts with Theorem 1(ii), we conclude that $f \in B_{p,q}^{s,w}$, and we obtain also the equivalence of quasi-norms. The proof of the corollary is complete.

We now turn to the proof of Proposition 7. First, we observe that the necessary parts follow easily from Corollary 10. Next, consider the sufficient part of (i) in the case $k = 0$. For simplicity, let

$$I(u) = \left(\int_0^T (t^{-s/2} \|u(\cdot, t)\|_{p,w})^q t^{-1} \, dt \right)^{1/q}.$$

Take $0 < \rho < \min(p, q)$ such that $r = p/\rho > r_0$. For $x \in R^n$ and $0 < t < T$, we derive from Lemma 9 and Hölder's inequality that

$$|u(x, t)|^\rho \leqslant Ct^{-n/2 + \rho s/2} I(u)^\rho \left(\int_{\{|z-x| \leqslant \sqrt{nt}\}} w(z)^{-r'/r} \, dz \right)^{1/r'},$$

where $1/r + 1/r' = 1$. Since $w^{-r'/r} \in A_{r'}$ $(w \in A_r)$, it follows from the (B_r)-condition (cf. [5, p. 582]) that

$$|u(x, t)|^\rho \leqslant CI(u)^\rho t^{-n/2 + \rho s/2} (1 + t)^{n/2} (1 + |x|)^n, \tag{29}$$

which implies, by Lemma 11, that

$$u(x, t_1 + t_2) = W_{t_1} * u(\cdot, t_2)(x) \tag{30}$$

for $x \in R^n$, $t_1, t_2 > 0$ and $t_1 + t_2 < T$. Define

$$\begin{aligned} v(x, t) &= u(x, t) && \text{if} \quad 0 < t \leqslant T/2, \\ &= W_{t-T/2} * u(\cdot, T/2)(x) && \text{if} \quad t > T/2. \end{aligned}$$

Then, it is easily seen that v is a temperature on R_+^{n+1}, and $v = u$ on D_T by (30). An easy computation with the aid of (29) yields

$$|v(x, t)| \leqslant Ct^{s/2 - n/2\rho} (1 + t)^{-s/2 + n/\rho} (1 + |x|)^{n/\rho}$$

for all x and t, where C might depend on u, T, and s. Therefore, it follows from a result of Flett [8, Theorem 17] that there exists a distribution f such that v is the Gauss–Weierstrass integral of f on R^{n+1}_{+}. Since $u=v$ on D_T, Theorem 1(ii) implies that $f \in B^{s,w}_{p,q}$, and hence we obtain the desired result in the case $k=0$. The sufficient part of (i) in the case $k>0$ is deduced by combining the technique used in the proof of Corollary 10 with that used in the proof for the case $k=0$.

The sufficient part of (ii) is proved by a reasoning similar to the proof of (i), so that details are omitted.

LEMMA 11. *Let u be a temperature on D_T. Assume that there exist a, b, and d for which*

$$|u(x,t)| \leqslant Ct^a(1+t)^b(1+|x|)^d, \qquad x \in R^n, \quad 0<t<T.$$

Then, for any $t_1, t_2>0$, $t_1+t_2<T$, and $x \in R^n$,

$$u(x, t_1+t_2) = W_{t_1} * u(\cdot, t_2)(x).$$

Proof. The desired result is an easy consequence of a uniqueness criterion for temperatures (cf. [8, Lemma 5 and Proof of Theorem 4]).

3.5. *Outlines of the Proofs of Theorems* 1′ *and* 4′

The proofs of the sufficient parts are almost identical to those for the nonhomogeneous spaces with proper notational changes. We notice that in the proofs of the necessary parts in the nonhomogeneous case, we have used the representation

$$f = \Psi * f + \sum_{j=1}^{\infty} \psi_j * f \qquad \text{in } \mathscr{S}'.$$

Since no information on $\Psi * f$ is available in the homogeneous case, we try to obtain a similar representation of f with the aid of $\psi_j * f$, $j=0, \pm 1, \pm 2, \ldots$. In fact, for an arbitrary distribution f, the discussion given by Peetre [14, pp. 51–56] yields polynomials P and P_m, $m=1,2,\ldots$, for which

$$f - P = \lim_{m \to \infty} \left(\sum_{j=-m}^{\infty} \psi_j * f + P_m \right) \qquad \text{in } \mathscr{S}', \tag{31}$$

where the degree of each P_m is less than N_f, a nonnegative integer that might depend on f; we assume that a polynomial of degree less than zero is the zero polynomial. Now, if $f \in \dot{B}^{s,w}_{p,\infty}$ (which contains both $\dot{B}^{s,w}_{p,q}$ and $\dot{F}^{s,w}_{p,q}$ for any q), then $f \in \dot{B}^{s-d/p}_{\infty,\infty}$ by [5, Theorem 2.6(iv) and Remark 2.7], and hence the argument of Peetre quoted above implies that we can take $N_f \leqslant \max(0, s-d/p)$. Consequently,

$$(\partial/\partial t)^k W_t * P_m = W_t * (-\Delta)^k P_m = 0 \qquad \text{for each } m$$

$(k > s/2)$, and we derive from (31) that

$$(\partial/\partial t)^k W_t * (f - P) = \lim_{m \to \infty} (\partial/\partial t)^k W_t * g_m, \tag{32}$$

$$\Psi_\rho * |(\partial/\partial t)^k W_t * (f - P)| = \lim_{m \to \infty} \Psi_\rho * |(\partial/\partial t)^k W_t * f_m|, \qquad 0 < \rho < \infty, \tag{33}$$

pointwise, where we set $f_m = \sum_{j=-m}^{\infty} \psi_j * f = g_m = \sum_{j=-m}^{\infty} \psi_j * \phi_j * f$ (ϕ_j are the functions in the proof of Theorem 4(i)). Inequality (10) then follows from (32) by an argument similar to that used in the estimate of V_λ^* in the proof of Theorem 4(i). As for the $\dot{B}$-space case, we observe from the homogeneous version of [5, Theorem 2.2] that

$$\|f_m\|_{\dot{B}(s, w; p, q)} \leqslant C \|f\|_{\dot{B}(s, w; p, q)}, \tag{34}$$

where C is independent of f and m. On the other hand, (33) gives

$$\begin{aligned} &\sup_{0<\rho<\infty} |\Psi_\rho * |(\partial/\partial t)^k W_t * (f - P)|(x)| \\ &\quad \leqslant \liminf_{m \to \infty} \{ \sup_{0<\rho<\infty} |\Psi_\rho * |(\partial/\partial t)^k W_t * f_m|(x)| \} \end{aligned}$$

for any $x \in R^n$, and thus,

$$\|(\partial/\partial t)^k W_t * (f - P)\|_{H(p, w)} \leqslant \liminf_{m \to \infty} \|(\partial/\partial t)^k W_t * f_m\|_{H(p, w)}$$

by Fatou's lemma. The desired result (8) follows from another application of Fatou's lemma and (34) provided we can prove

$$\left(\int_0^\infty (t^{k - s/2} \|(\partial/\partial t)^k W_t * f\|_{H(p, w)})^q \, t^{-1} \, dt \right)^{1/q} \leqslant C \|f\|_{\dot{B}(s, w; p, q)} \tag{35}$$

for any $f \in \dot{B}_{p,q}^{s,w}$ such that $\hat{f} = 0$ in a neighborhood of the origin; notice that each f_m satisfies this last assumption. Under this additional assumption on f, we see that

$$f = \sum_{j=-\infty}^{\infty} \psi_j * f \qquad \text{in } \mathscr{S}',$$

and the proof of the nonhomogeneous case (Theorem 1(i)) can be carried over with minor modifications. The two most nontrivial changes needed in the proof of (35) are to replace the Bessel potential operators by the Riesz potential operators and to perform interpolation on subspaces of homogeneous Besov spaces consisting of distributions whose Fourier transforms vanish on a neighborhood of the origin. We observe that in the proof of (35) (under the assumption $\hat{f}=0$ in a neighborhood of the origin), we need not assume $w \in \dot{\mathscr{M}}_d$.

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